Lecture 2
Camera Models

Professor Silvio Savarese
Computational Vision and Geometry Lab
Announcements

Prerequisites: any questions?

This course requires knowledge of linear algebra, probability, statistics, machine learning and computer vision, as well as decent programming skills. Though not an absolute requirement, it is encouraged and preferred that you have at least taken either CS221 or CS229 or CS131A or have equivalent knowledge.

Topics such as **linear filters, feature detectors and descriptors, low level segmentation, tracking, optical flow, clustering and PCA/LDA techniques** for recognition won’t be covered in CS231.

We will provide links to background material related to CS131A (or discuss during TA sessions) so students can refresh or study those topics if needed.

We will leverage concepts from machine learning (CS229) (e.g., **SVM, basic Bayesian inference, clustering**, etc...) which we won’t cover in this class either. Again, we will supply links to related material for background reading.
Announcements

Next TA session: Fridays from 2:15-3:05pm
Lecture 2

Camera Models

• Pinhole cameras
• Cameras & lenses
• The geometry of pinhole cameras
• Other camera models

Reading:

[FP] Chapter 1 “Cameras”
[FP] Chapter 2 “Geometric Camera Models”
[HZ] Chapter 6 “Camera Models”

Some slides in this lecture are courtesy to Profs. J. Ponce, S. Seitz, F-F Li
How do we see the world?

- Let’s design a camera
  - Idea 1: put a piece of film in front of an object
  - Do we get a reasonable image?
Pinhole camera

- Add a barrier to block off most of the rays
  - This reduces blurring
  - The opening known as the aperture
Some history...

Milestones:
• Leonardo da Vinci (1452-1519): first record of camera obscura
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Photography (Niepce, “La Table Servie,” 1822)
Some history...

Milestones:
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• Daguerréotypes (1839)
• Photographic Film (Eastman, 1889)
• Cinema (Lumiére Brothers, 1895)
• Color Photography (Lumiére Brothers, 1908)

Photography (Niepce, “La Table Servie,” 1822)
Let’s also not forget…

Motzu (468-376 BC)
Oldest existent book on geometry in China

Aristotle (384-322 BC)
Also: Plato, Euclid

Al-Kindi (c. 801–873)
Ibn al-Haitham (965-1040)
f = focal length
o = aperture = pinhole = center of the camera
Pinhole camera

\[ P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \]

\[
\begin{align*}
x' &= f \frac{x}{z} \\
y' &= f \frac{y}{z}
\end{align*}
\]

Derived using similar triangles
Pinhole camera

\[ P = [x, z] \]

\[ P' = [x', f] \]

\[ \frac{x'}{f} = \frac{x}{z} \]
Common to draw image plane in front of the focal point.

What's the transformation between these 2 planes?

\[
\begin{align*}
x' &= f \frac{x}{z} \\
y' &= f \frac{y}{z}
\end{align*}
\]
Pinhole camera

Is the size of the aperture important?
Shrinking aperture size

- Rays are mixed up

- Why the aperture cannot be too small?
  - Less light passes through
  - Diffraction effect

Adding lenses!
Cameras & Lenses

- A lens focuses light onto the film
A lens focuses light onto the film
- There is a specific distance at which objects are “in focus”
- Related to the concept of depth of field
Cameras & Lenses

• A lens focuses light onto the film
  – There is a specific distance at which objects are “in focus”
  – Related to the concept of depth of field
A lens focuses light onto the film

- All parallel rays converge to one point on a plane located at the focal length $f$
- Rays passing through the center are not deviated
Thin Lenses

Snell’s law:

\[ n_1 \sin \alpha_1 = n_2 \sin \alpha_2 \]

Small angles:

\[ n_1 \alpha_1 \approx n_2 \alpha_2 \]

- \( n_1 = n \) (lens)
- \( n_1 = 1 \) (air)

Focal length:

\[ f = \frac{R}{2(n - 1)} \]

\[ z' = f + z_0 \]

For details see lecture on cameras in CS131A
Issues with lenses: Radial Distortion

Pin cushion

Barrel (fisheye lens)
Issues with lenses: Radial Distortion

- Deviations are most noticeable for rays that pass through the edge of the lens

No distortion

Pin cushion

Barrel (fisheye lens)

Image magnification decreases with distance from the optical axis
Lecture 2
Camera Models

• Pinhole cameras
• Cameras & lenses
• The geometry of pinhole cameras
  • Intrinsic
  • Extrinsic
• Other camera models
Pinhole camera

\[ f = \text{focal length} \]
\[ o = \text{center of the camera} \]

\[ (x, y, z) \rightarrow (f \frac{x}{z}, f \frac{y}{z}) \]

\[ \mathbb{R}^3 \rightarrow \mathbb{R}^2 \]
From retina plane to images

Pixels, bottom-left coordinate systems
Coordinate systems
Converting to pixels

1. Off set

\[(x, y, z) \rightarrow \left( f \frac{x}{z} + c_x, f \frac{y}{z} + c_y \right)\]

\(C = [c_x, c_y]\)
Converting to pixels

1. Off set
2. From metric to pixels

\[(x, y, z) \rightarrow (f \frac{k}{z}x + c_x, f \frac{l}{z}y + c_y)\]

Units: \(k, l: \text{pixel/m} \quad \alpha, \beta: \text{pixel}\)

Non-square pixels

\[C = [c_x, c_y]\]
Converting to pixels

\[(x, y, z) \rightarrow (\alpha \frac{x}{z} + c_x, \beta \frac{y}{z} + c_y)\]

- Matrix form?

A related question:
- Is this a linear transformation?
(x, y, z) → (f \frac{x}{z}, f \frac{y}{z})

Is this a linear transformation?

No — division by z is nonlinear

How to make it linear?
Homogeneous coordinates

\[(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\]

homogeneous image coordinates

\[(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}\]

homogeneous scene coordinates

• Converting \textit{from} homogeneous coordinates

\[
\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)
\]

\[
\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)
\]
Camera Matrix

\[(x, y, z) \rightarrow (\alpha \frac{x}{z} + c_x, \beta \frac{y}{z} + c_y)\]

\[
X' = \begin{bmatrix}
\alpha x + c_x z \\
\beta y + c_y z \\
z
\end{bmatrix} = \begin{bmatrix}
\alpha & 0 & c_x & 0 \\
0 & \beta & c_y & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]
Perspective Projection Transformation

\[
X' = \begin{bmatrix}
    f & x \\
    f & y \\
    z & \end{bmatrix} = \begin{bmatrix}
    f & 0 & 0 & 0 \\
    0 & f & 0 & 0 \\
    0 & 0 & 1 & 0 \\
\end{bmatrix} \begin{bmatrix}
    x \\
    y \\
    z \\
    1 \\
\end{bmatrix}
\]

\[
X' = M X
\]

\[
\mathbb{R}^4 \xrightarrow{H} \mathbb{R}^3
\]
\[ \mathbf{X'} = \mathbf{M} \mathbf{X} = \mathbf{K} \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{X} \]
Finite projective cameras

\[
X' = \begin{bmatrix}
\alpha & s & c_x & 0 \\
0 & \beta & c_y & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]

K has 5 degrees of freedom!

Skew parameter C=[c_x, c_y]
Lecture 2

Camera Models

- Pinhole cameras
- Cameras & lenses
- The geometry of pinhole cameras
  - Intrinsic
  - Extrinsic
- Other camera models
The mapping so far is defined within the camera reference system.

What if an object is represented in the world reference system?
2D Translation

For details see lecture on transformations in CS131A
See also TA session on Friday
2D Translation Equation

\[ P' = P + t = (x + t_x, y + t_y) \]

\[ P = (x, y) \]
\[ t = (t_x, t_y) \]
2D Translation using Homogeneous Coordinates

\[ P = (x, y) \rightarrow (x, y, 1) \]
\[ t = (t_x, t_y) \rightarrow (t_x, t_y, 1) \]

\[ P' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = [I \ t] \cdot P = T \cdot P \]
Scaling Equation

\[ \mathbf{P} = (x, y) \rightarrow \mathbf{P}' = (s_x x, s_y y) \]

\[ \mathbf{P} = (x, y) \rightarrow (x, y, 1) \]

\[ \mathbf{P}' = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1) \]

\[ \mathbf{P}' = \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} S' & 0 \\ 0 & 1 \end{bmatrix} \cdot \mathbf{P} = \mathbf{S} \cdot \mathbf{P} \]
Scaling & Translating

\[
P'' = T \cdot P' = T \cdot (S \cdot P) = T \cdot S \cdot P = A \cdot P
\]
Scaling & Translating

\[
P'' = T \cdot S \cdot P = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\
= \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}
\]

\[
A = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix}
\]
Rotation
Rotation Equations

- Counter-clockwise rotation by an angle $\theta$

\[
x' = \cos \theta x - \sin \theta y
\]
\[
y' = \cos \theta y + \sin \theta x
\]

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}
\]

\[P' = R \cdot P\]
Degrees of Freedom

\[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} = \begin{bmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]

\[R\text{ is 2x2} \quad \Rightarrow \quad 4\text{ elements}\]

Note: \(R\) is an orthogonal matrix and satisfies many interesting properties:

\[
R \cdot R^T = R^T \cdot R = I
\]

\[
\det(R) = 1
\]
Rotation + Scale + Translation

\[
P' = (T \cdot R \cdot S) \cdot P
\]

\[
P' = T \cdot R \cdot S \cdot P = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} \cos \theta & -\sin \theta & t_x \\ \sin \theta & \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} R' & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} R' & S & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
\]

If \( s_x = s_y \), this is a similarity transformation!
3D Rotation of Points

Rotation around the coordinate axes, counter-clockwise:

\[
R_x (\alpha) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha \\
0 & \sin \alpha & \cos \alpha
\end{bmatrix}
\]

\[
R_y (\beta) = \begin{bmatrix}
\cos \beta & 0 & \sin \beta \\
0 & 1 & 0 \\
-\sin \beta & 0 & \cos \beta
\end{bmatrix}
\]

\[
R_z (\gamma) = \begin{bmatrix}
\cos \gamma & -\sin \gamma & 0 \\
\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
World reference system

In 4D homogeneous coordinates:

\[ X' = K[I \ 0]X = K[I \ 0]\begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}_{4 \times 4} X_w = K[R \ T]X_w \]
Projective cameras

\[ X'_{3 \times 1} = M_{3 \times 4} \quad X_w = K_{3 \times 3} [R \quad T]_{3 \times 4} \quad X_w^{4 \times 1} \]

\[ K = \begin{bmatrix} \alpha & s & c_x \\ 0 & \beta & c_y \\ 0 & 0 & 1 \end{bmatrix} \]

How many degrees of freedom?

\[ 5 + 3 + 3 = 11! \]
Projective cameras

\[
X'_{3 \times 1} = M X_w = K_{3 \times 3} \begin{bmatrix} R & T \end{bmatrix}_{3 \times 4} X_w_{4 \times 1}
\]

\[
M = \begin{bmatrix}
  m_1 \\
  m_2 \\
  m_3 \\
\end{bmatrix}
\]

\[
E \rightarrow \left( \frac{m_1 X_w}{m_3 X_w}, \frac{m_2 X_w}{m_3 X_w} \right)
\]
Theorem (Faugeras, 1993)

\[ M = K [R \quad T] = [K R \quad KT] = [A \quad b] \]

\[ K = \begin{bmatrix}
\alpha & s & c_x \\
0 & \beta & c_y \\
0 & 0 & 1
\end{bmatrix} \quad \alpha = f \, k; \quad \beta = f \, l \quad A = \begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix} \]

- A necessary and sufficient condition for \( M \) to be a perspective projection matrix is that \( \text{Det}(A) \neq 0 \).
- A necessary and sufficient condition for \( M \) to be a zero-skew perspective projection matrix is that \( \text{Det}(A) \neq 0 \) and
  \[ (a_1 \times a_3) \cdot (a_2 \times a_3) = 0. \]
- A necessary and sufficient condition for \( M \) to be a perspective projection matrix with zero skew and unit aspect-ratio is that \( \text{Det}(A) \neq 0 \) and
  \[ \begin{cases}
  (a_1 \times a_3) \cdot (a_2 \times a_3) = 0, \\
  (a_1 \times a_3) \cdot (a_1 \times a_3) = (a_2 \times a_3) \cdot (a_2 \times a_3).
\end{cases} \]
Properties of Projection

• Points project to points
• Lines project to lines
• Distant objects look smaller
Properties of Projection

• Angles are not preserved
• Parallel lines meet!

Vanishing point
One-point perspective

• Masaccio, *Trinity*, Santa Maria Novella, Florence, 1425-28
Next lecture

• How to calibrate a camera?