Problem Set 2 Review

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Outline

- Block matrix multiplication
- 8-point algorithm
- Factorization

Block matrix multiplication

Block matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1s} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{q1} & \mathbf{A}_{q2} & \cdots & \mathbf{A}_{qs} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \cdots & \mathbf{B}_{1r} \\ \mathbf{B}_{21} & \mathbf{B}_{22} & \cdots & \mathbf{B}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{s1} & \mathbf{B}_{s2} & \cdots & \mathbf{B}_{sr} \end{bmatrix}$$

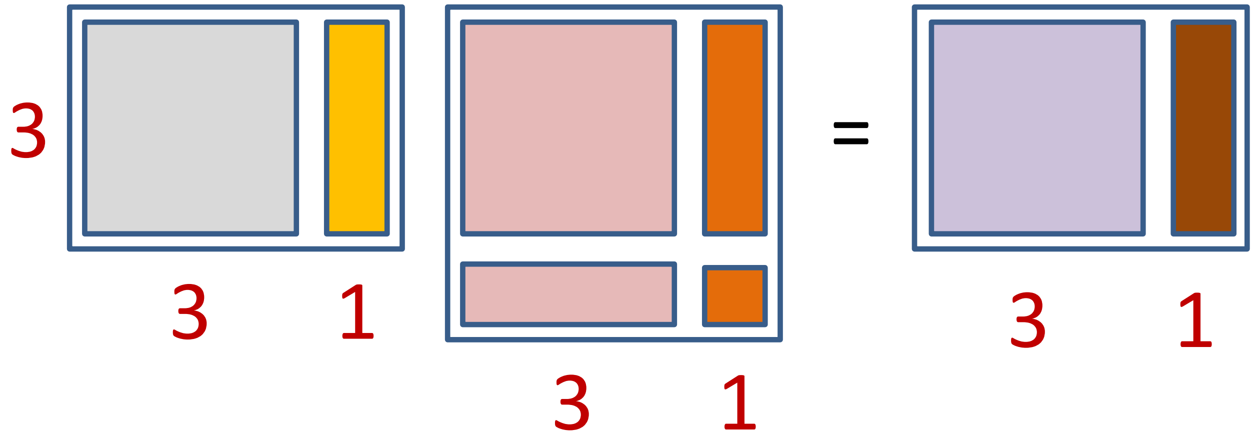
$$\mathbf{C} = \mathbf{A}\mathbf{B}$$

$$\mathbf{C}_{\alpha\beta} = \sum_{\gamma=1}^{s} \mathbf{A}_{\alpha\gamma}\mathbf{B}_{\gamma\beta}.$$

Just treat them as elements.

Problem 1

- $MH = [A, b]\begin{bmatrix} H_1, H_2 \\ H_3, H_4 \end{bmatrix} = [I_3, 0]$



→$AH_1 + bH_3 = I_3$

→$AH_2 + bH_4 = 0$

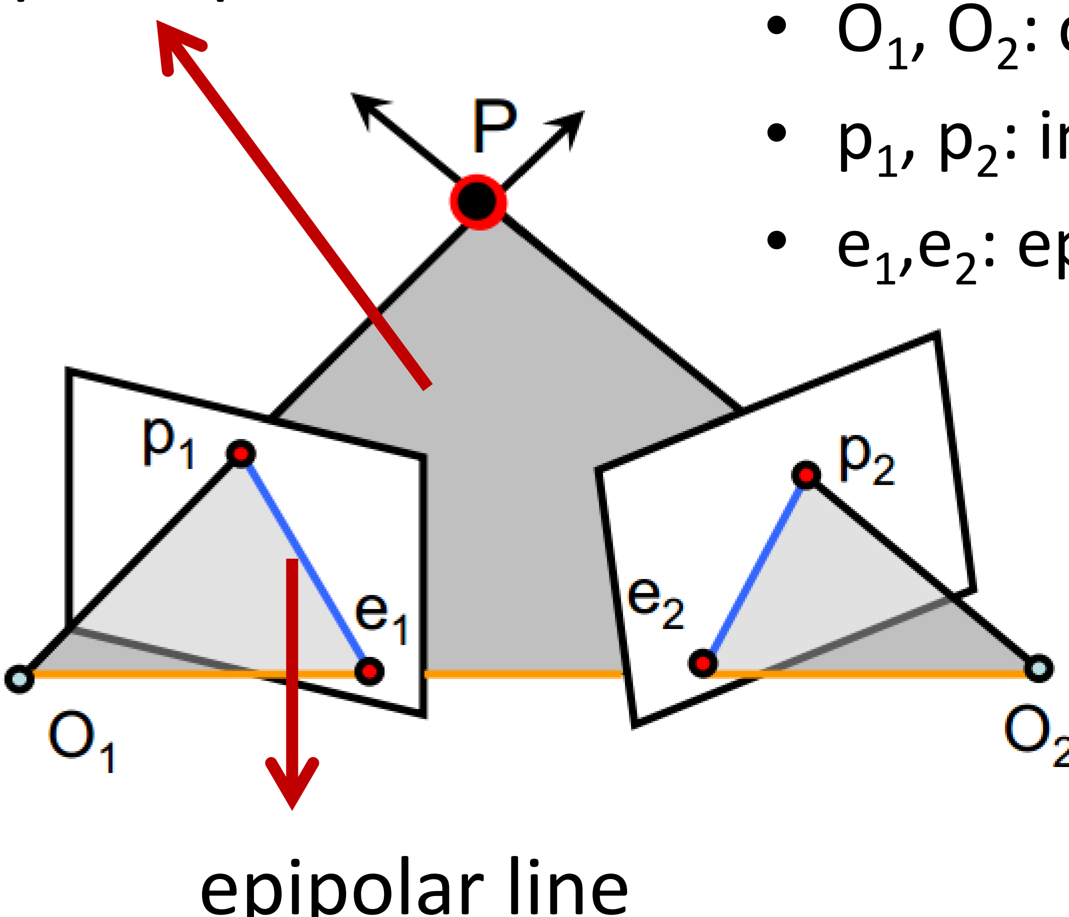
How to choose H_3 and H_4?

8-point algorithm

Epipolar geometry

- P: object
- O_1, O_2: center of camera
- p_1, p_2: image point
- e_1,e_2: epipole

epipolar plane



Fundamental matrix *F*

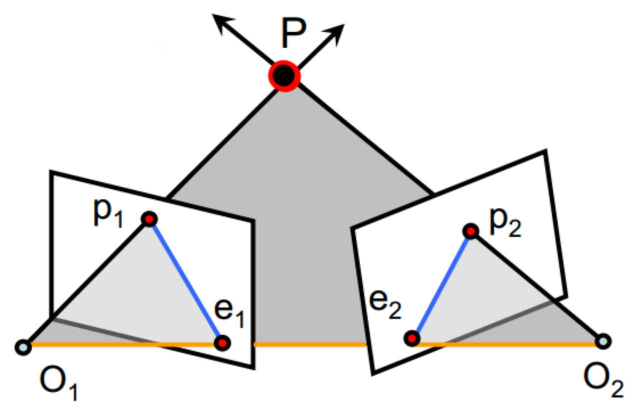
$$p_1^T \cdot F \; p_2 = 0$$

- *F* is rank 2
  - why? $F = K^{-T}[T_\times]RK'^{-1}$, and $T_\times$ is rank 2.
  - Use SVD to ensure this property.
- *F* has 7 dof
  - 8 independent ratio due to scaling.
  - $\det F = 0 \rightarrow$ 7 dof
- Transpose
  - *F* for cameras (O_1, O_2) iff F^T for cameras (O_2, O_1)

Fundamental matrix *F* (cont'd)

$$p_1^T \cdot F\ p_2 = 0$$

- Epipolar lines: $l_1 = F p_2, \;\; p_1^T \cdot l_1 = 0$
  - 2D line: $\bar{x} \cdot \tilde{l} = ax + by + c = 0.$
- Epipole: $\forall p_2, \;\; e_1^T (F p_2) = 0$
  - e_1 is left null vector of *F*
  - Similarly, $\forall p_1, (p_1^T F) e_2 = 0,$
    so e_2 is right null vector of *F*



- Correlation: for epipolar line pair *l* and *l*', any point *p* on *l* is mapped to *l*' (no inverse)

Computation of *F*

$$p_1^T \cdot F\ p_2 = 0$$

$$(u, v, 1)\begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}\begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

For each pair of corresponding points (u', v', 1), (u, v, 1):

$$(uu', uv', u, vu', vv', v, u', v', 1)\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

8-point algorithm!

Numerical error

$$\begin{bmatrix} u_1u_1' & u_1v_1' & u_1 & v_1u_1' & v_1v_1' & v_1 & u_1' & v_1' & 1 \\ u_2u_2' & u_2v_2' & u_2 & v_2u_2' & v_2v_2' & v_2 & u_2' & v_2' & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_nu_n' & u_nv_n' & u_n & v_nu_n' & v_nv_n' & v_n & u_n' & v_n' & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

~10000 ~10000 ~100 ~10000 ~10000 ~100 ~100 ~100 1

! Orders of magnitude difference between column of data matrix
$\rightarrow$ least-squares yields poor results

Normalized 8-point algorithm

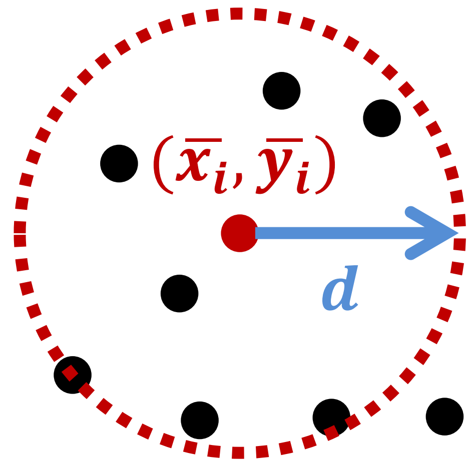
- Normalize: $q_i = Tp_i, \;\; q_i' = T'p_i{}'$
- 8-point algorithm to solve $F_q{}'$ from → **SVD!**

$$q_i'^T F_q' q_i = 0$$

- Force $F_q{}'$ to have rank 2 → **SVD!**
- De-normalize F_q to get F

$$F = T'^T F_q T$$

Normalizing data points



- Goal
  - Mean: 0
  - Average distance to the mean: $\sqrt{2}$
- Intuitively, we want $q_i = (p_i - \overline{p_i})\frac{\sqrt{2}}{d}$
  - $\overline{x_i} = \frac{1}{n}\sum_i x_i,\ \overline{y_i} = \frac{1}{n}\sum_i y_i,$
  - $d = \frac{1}{n}\sum_i \sqrt{(x_i - \overline{x_i})^2 + (y_i - \overline{y_i})^2}$
- $q_i = \begin{bmatrix} \sqrt{2}/d & 0 & -\bar{x}\sqrt{2}/d \\ 0 & \sqrt{2}/d & -\bar{y}\sqrt{2}/d \\ 0 & 0 & 1 \end{bmatrix} p_i$

3x1 **3x1**

Use SVD on least square problem

- Solve over-determined $Ax = 0$

$$\min |Ax|^2$$
$$\text{s.t.} |x|^2 = 1$$

From SVD, $A = U\Sigma V^T$, want to minimize

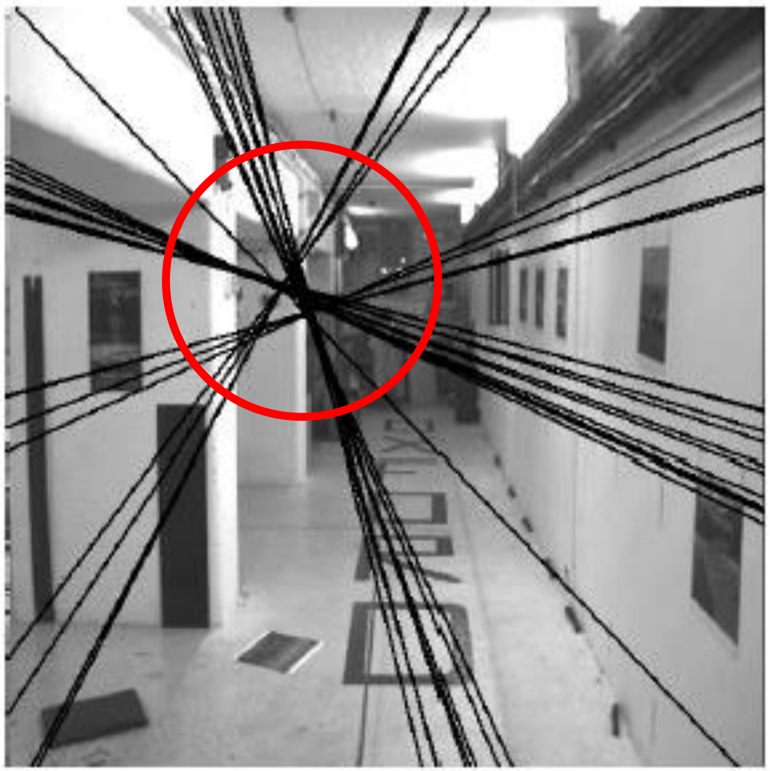
$$\begin{aligned} &|Ax|^2 \\ &= x^T A^T A x \\ &= x^T (U\Sigma V^T)^T (U\Sigma V) x \\ &= x^T V \Sigma^T U^T U \Sigma V^T x \\ &= x^T V \Sigma^T \Sigma V^T x \\ &= \sum_k \sigma_k^2 \left(v_k^T x\right)^2 \end{aligned}$$

Choose *x* to be v_k corresponding to smallest σ_k

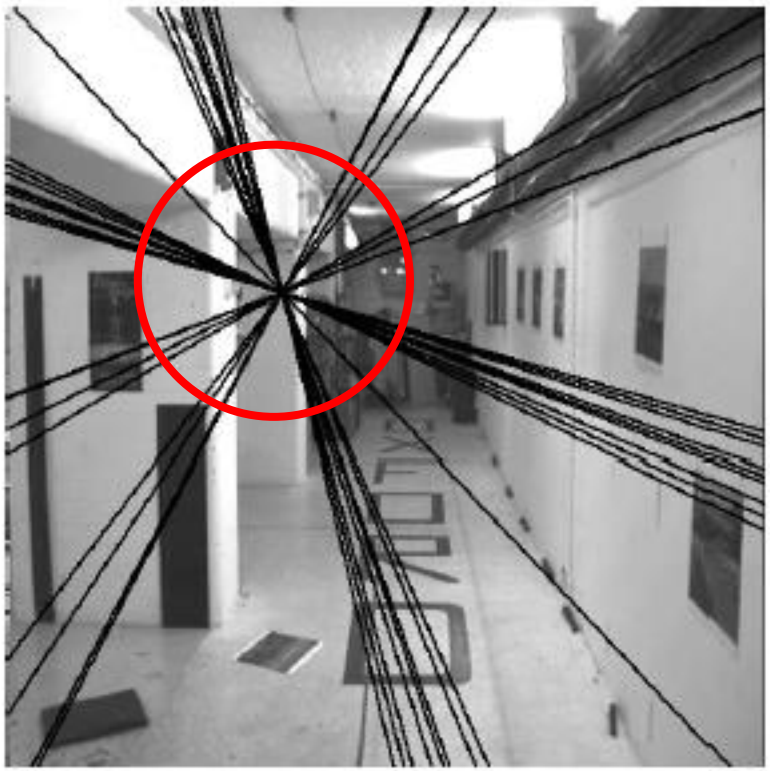
Use SVD to reduce rank

- $A = U\Sigma V^T = U \begin{bmatrix} \sigma_1 & \cdots & \cdots \\ \vdots & \sigma_2 & \vdots \\ \cdots & \cdots & \ddots \end{bmatrix} V^T = \sum_i \sigma_i u_i v_i^T$
- Intuition: only retain *k* components
  - Gives best rank *k* approximation of *A*
- For formal proof, see Eckart-Young theorem

Enforcing rank 2 on *F*



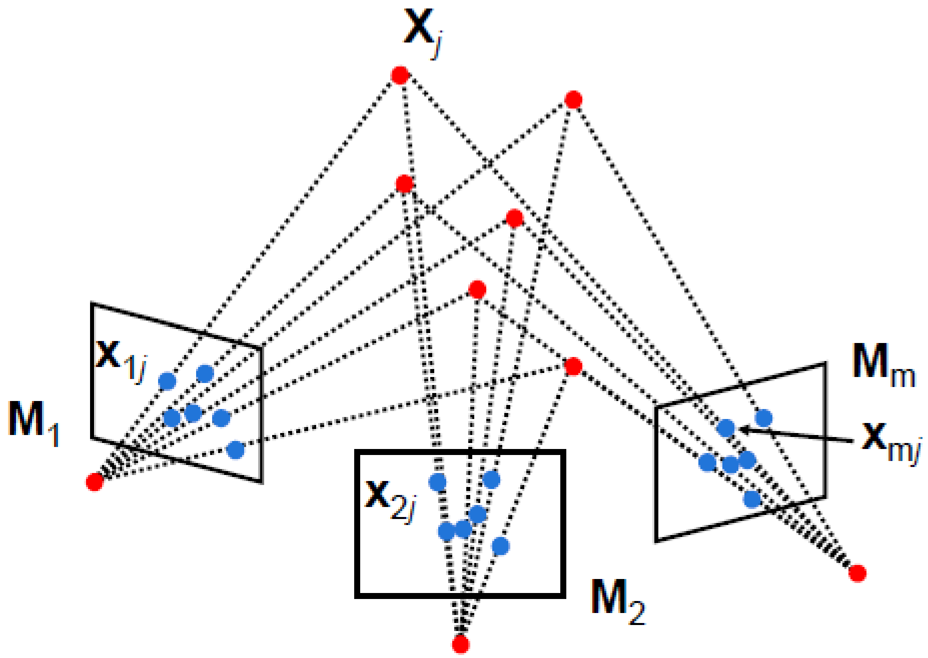
Non-singular *F*



Singular *F*

Factorization

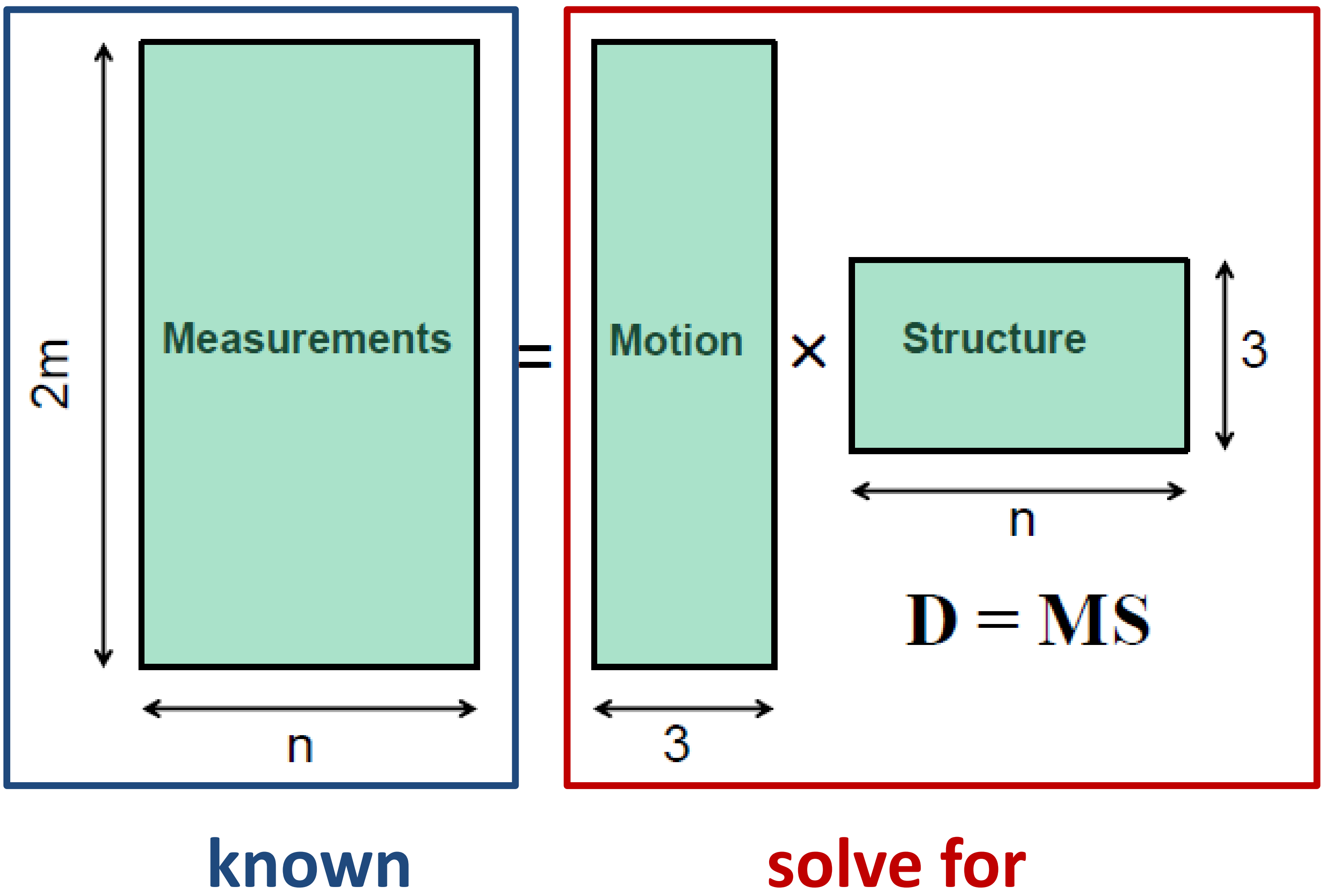
Structure From Motion



$$\underline{\mathbf{x}_{ij}} = \underline{\mathbf{M}_i \mathbf{X}_j}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$$

known **solve for**

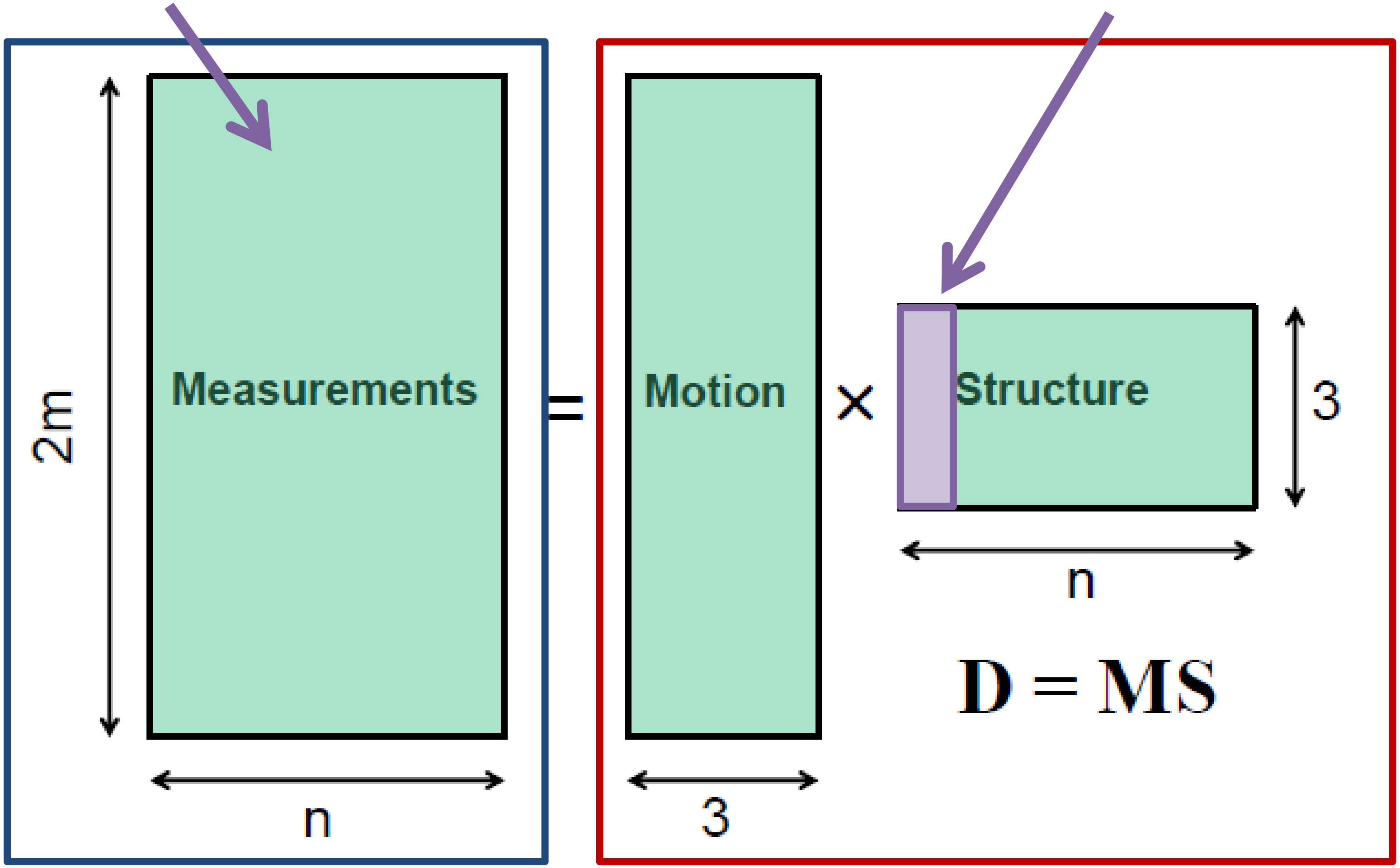
Factorization



Factorization

(1) $\hat{\mathbf{x}}_{ij} = \mathbf{x}_{ij} - \frac{1}{n}\sum_{k=1}^{n} \mathbf{x}_{ik}$

(3) Columns are the 3D points



(2) SVD

Factorization

- DEMO