Outline

• Block matrix multiplication
• 8-point algorithm
• Factorization
Block matrix multiplication
Block matrix

\[ A = \begin{bmatrix}
    A_{11} & A_{12} & \cdots & A_{1s} \\
    A_{21} & A_{22} & \cdots & A_{2s} \\
    \vdots & \vdots & \ddots & \vdots \\
    A_{q1} & A_{q2} & \cdots & A_{qs}
\end{bmatrix} \quad B = \begin{bmatrix}
    B_{11} & B_{12} & \cdots & B_{1r} \\
    B_{21} & B_{22} & \cdots & B_{2r} \\
    \vdots & \vdots & \ddots & \vdots \\
    B_{s1} & B_{s2} & \cdots & B_{sr}
\end{bmatrix} \]

\[ C = AB \]

\[ C_{\alpha\beta} = \sum_{\gamma=1}^{s} A_{\alpha\gamma} B_{\gamma\beta}. \]

Just treat them as elements.
Problem 1

- $MH = [A, b] \begin{bmatrix} H_1, H_2 \\ H_3, H_4 \end{bmatrix} = [I_3, 0]$

$\Rightarrow AH_1 + bH_3 = I_3$

$\Rightarrow AH_2 + bH_4 = 0$

How to choose $H_3$ and $H_4$?
8-point algorithm
Epipolar geometry

- $P$: object
- $O_1, O_2$: center of camera
- $p_1, p_2$: image point
- $e_1, e_2$: epipole

epipolar plane

epipolar line
Fundamental matrix $F$

$$p_1^T \cdot F \cdot p_2 = 0$$

- $F$ is rank 2
  - why? $F = K^{-T} [T_x] R K'^{-1}$, and $T_x$ is rank 2.
  - Use SVD to ensure this property.

- $F$ has 7 dof
  - 8 independent ratio due to scaling.
  - $\det F = 0 \rightarrow 7$ dof

- Transpose
  - $F$ for cameras $(O_1, O_2)$ iff $F^T$ for cameras $(O_2, O_1)$
Fundamental matrix $F$ (cont'd)

\[ p_1^T \cdot F \cdot p_2 = 0 \]

• Epipolar lines: \( l_1 = Fp_2, \quad p_1^T \cdot l_1 = 0 \)
  – 2D line: \( \bar{x} \cdot \bar{l} = ax + by + c = 0. \)

• Epipole: \( \forall p_2, \quad e_1^T (Fp_2) = 0 \)
  – \( e_1 \) is left null vector of \( F \)
  – Similarly, \( \forall p_1, \quad (p_1^T F)e_2 = 0, \)
    so \( e_2 \) is right null vector of \( F \)

• Correlation: for epipolar line pair \( l \) and \( l' \), any point \( p \) on \( l \) is mapped to \( l' \) (no inverse)
Computation of $F$

$$p_1^T \cdot F \cdot p_2 = 0$$

$$(u, v, 1) \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

For each pair of corresponding points $(u', v', 1), (u, v, 1)$:

$$\begin{pmatrix} u u' , u v' , u, v u' , v v' , v, u', v', 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

8-point algorithm!
Numerical error

Orders of magnitude difference between column of data matrix → least-squares yields poor results
Normalized 8-point algorithm

• Normalize: \( q_i = T p_i, \ q'_i = T' p'_i \)

• 8-point algorithm to solve \( F \) from
  \[ q_i^T F_q q'_i = 0 \]

• Force \( F_q \) to have rank 2

• De-normalize \( F_q \) to get \( F \)
  \[ F = T'^T F_q T' \]
Normalizing data points

- **Goal**
  - Mean: 0
  - Average distance to the mean: $\sqrt{2}$

- **Intuitively, we want** $q_i = (p_i - \bar{p}_i) \frac{\sqrt{2}}{d}$

- $\bar{x}_i = \frac{1}{n} \sum_i x_i$, $\bar{y}_i = \frac{1}{n} \sum_i y_i$,
- $d = \frac{1}{n} \sum_i \sqrt{(x_i - \bar{x}_i)^2 + (y_i - \bar{y}_i)^2}$

- $q_i = \begin{bmatrix} \sqrt{2}/d & 0 & -\bar{x}\sqrt{2}/d \\ 0 & \sqrt{2}/d & -\bar{y}\sqrt{2}/d \\ 0 & 0 & 1 \end{bmatrix} p_i$
Use SVD on least square problem

- Solve over-determined $Ax = 0$
  $$\begin{align*}
  \min & \ |Ax|^2 \\
  \text{s.t.} & \ |x|^2 = 1
  \end{align*}$$

From SVD, $A = U\Sigma V^T$, want to minimize
$$\begin{align*}
|Ax|^2 &= x^T A^T Ax \\
&= x^T (U\Sigma V^T)^T (U\Sigma V)x \\
&= x^T V\Sigma^T U^T U\Sigma V^T x \\
&= x^T V\Sigma^T \Sigma V^T x \\
&= \sum_k \sigma_k^2 (v_k^T x)^2
  \end{align*}$$

Choose $x$ to be $v_k$ corresponding to smallest $\sigma_k$
Use SVD to reduce rank

\[ A = U \Sigma V^T = U \begin{bmatrix} \sigma_1 & \cdots & \cdots \\ \vdots & \sigma_2 & \vdots \\ \cdots & \cdots & \cdots \end{bmatrix} V^T = \sum_i \sigma_i u_i v_i^T \]

• Intuition: only retain \( k \) components
  – Gives best rank \( k \) approximation of \( A \)

• For formal proof, see Eckart-Young theorem
Enforcing rank 2 on $F$

Non-singular $F$

Singular $F$
Factorization
Structure From Motion

\[ x_{ij} = M_i X_j, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \]

known \hspace{1cm} solve for
Factorization

\[
\text{Measurements} \times \text{Motion} = \text{Structure}
\]

\[
D = MS
\]

known

solve for
(1) \[ \hat{x}_{ij} = x_{ij} - \frac{1}{n} \sum_{k=1}^{n} x_{ik} \]

Factorization

(2) SVD

(3) Columns are the 3D points

\[ D = MS \]
Factorization

• DEMO