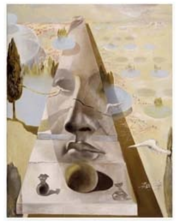
# Lecture 7
## Multi-view geometry



- The SFM problem
- Affine SFM
- Perspective SFM
- Self-calibration
- Applications

**Reading:**
[HZ] Chapter 10 "3D reconstruction of cameras and structure"
Chapter 18 "N-view computational methods"
Chapter 19 "Auto-calibration"
[FP] Chapter 13 "projective structure from motion"
[Szelisky] Chapter 7 "Structure from motion"



Today we'll continue studying the structure from motion problem. First, we'll recap the affine SFM problem, then we'll explore doing SFM with a more realistic assumption of perspective cameras and discuss the intrinsic ambiguities of SFM. In the second half of this lecture, we'll discuss the self-calibration problem (I.e. how to remove these ambiguities) and finally show some results.

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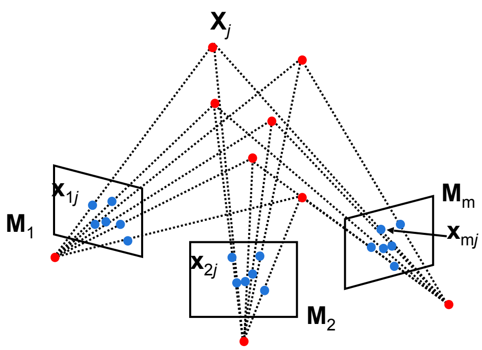
## Structure from motion problem




Given *m* images of *n* fixed 3D points

- $\mathbf{x}_{ij} = \mathbf{M}_i \mathbf{X}_j, \quad i = 1, \dots, m, \quad j = 1, \dots, n$

As a reminder, with have m cameras with projection matrices $M_i$, n points at world locations $X_j$, and m times n correspondences $x_{ij}$, recording the position of the j-th world point in the i-th image.

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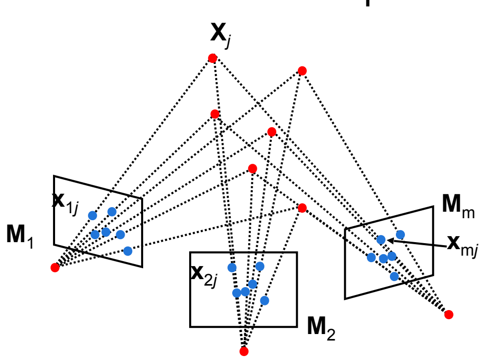
## Structure from motion problem



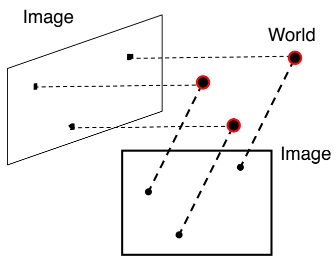

From the m×n correspondences $\mathbf{x}_{ij}$, estimate:

- *m* projection matrices $\mathbf{M}_i$ motion
- *n* 3D points $\mathbf{X}_j$ structure

We want to estimate both $M_i$, the poses of the cameras, and $X_j$, the locations of the points.

# Affine structure from motion

(simpler problem)

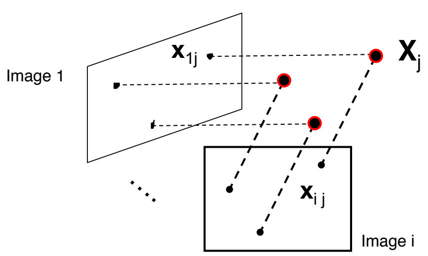



From the mxn correspondences $\mathbf{x}_{ij}$, estimate:

- $m$ projection matrices $\mathbf{M}_i$ (affine cameras)
- $n$ 3D points $\mathbf{X}_j$

We start with a simpler problem, assuming the cameras are affine or weak perspective.

# Affine cameras




Camera matrix **M** for the affine case

$$\mathbf{x}_{ij} = \mathbf{A}_i\mathbf{X}_j + \mathbf{b}_i = \mathbf{M}_i \begin{bmatrix} \mathbf{X}_j \\ 1 \end{bmatrix}; \qquad \mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$$

[Eq. 1]

We use the affine camera model to express the relationship from a point $X_j$ in 3D and the corresponding observations in each affine camera (for instance, $\mathbf{x}_{ij}$ in camera i). Notice that M has been redefined as [A b] (a 2x4 matrix) as opposed to the 3x4 form (see previous slide).

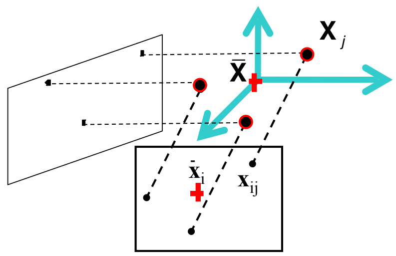
## A factorization method –
Tomasi & Kanade algorithm

C. Tomasi and T. Kanade[Shape and motion from image streams under orthography: A factorization method.](#) *IJCV*, 9(2):137-154, November 1992.

- Data centering
- Factorization

The factorization method can be used for solving the affine SFM problem. This method consists of two major steps: the data centering step and the actual factorization step.

## A factorization method - Centering the data




If the centroid of points in 3D = center of the world reference system

$$\hat{\mathbf{x}}_{ij} = \mathbf{A}_i \hat{\mathbf{X}}_j = \mathbf{A}_i \mathbf{X}_j \quad \text{[Eq. 2]}$$

By centering (normalizing) the observations $x_{ij}$ in each image, and by assuming that the world reference system is located at $X^{bar}$ (the location of the world reference system is arbitrary), we obtain from Eq.1 a very compact expression that relates (centered) observations and 3D points (Eq. 2).

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## A factorization method - factorization

Let's create a $2m \times n$ data (measurement) matrix:

$$\mathbf{D} = \underbrace{\begin{bmatrix} \hat{\mathbf{x}}_{11} & \hat{\mathbf{x}}_{12} & \cdots & \hat{\mathbf{x}}_{1n} \\ \hat{\mathbf{x}}_{21} & \hat{\mathbf{x}}_{22} & \cdots & \hat{\mathbf{x}}_{2n} \\ & & \ddots & \\ \hat{\mathbf{x}}_{m1} & \hat{\mathbf{x}}_{m2} & \cdots & \hat{\mathbf{x}}_{mn} \end{bmatrix}}_{(2m \times n)} = \underbrace{\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_m \end{bmatrix}}_{\text{cameras } (2m \times 3)\ \mathrm{M}} \underbrace{\begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \cdots & \mathbf{X}_n \end{bmatrix}}_{\text{points } (3 \times n)\ \mathrm{S}}$$

The measurement matrix **D = M S** has rank 3
(it's a product of a 2mx3 matrix and 3xn matrix)

D can be expressed as the product of the 2m×3 matrix M (which comprises the camera matrices $A_1 \ldots A_m$) and the 3×n matrix S (which comprises the 3D points $X_1, \ldots X_n$). The matrix D has rank three since it is the product of 2 matrices whose max dimension is 3.

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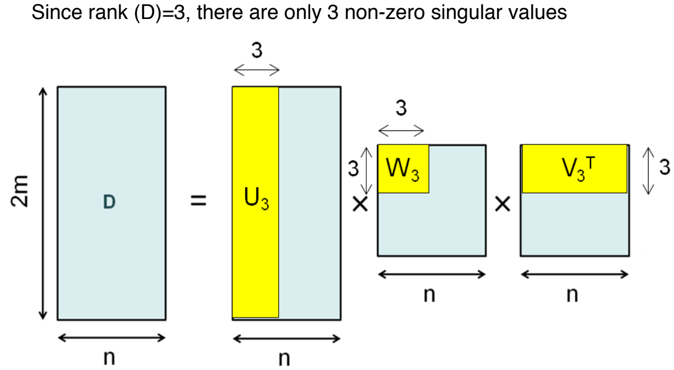
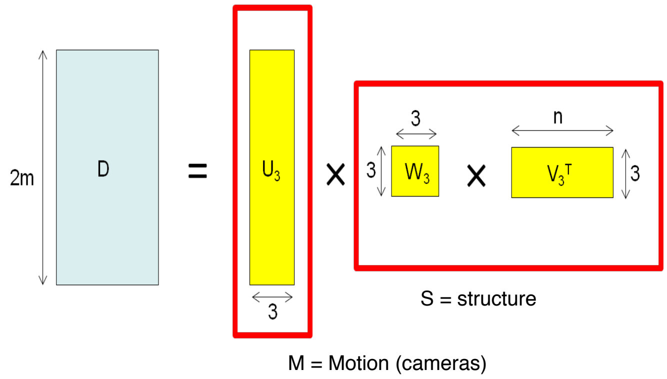
## Factorizing the Measurement Matrix by SVD

Since rank (D)=3, there are only 3 non-zero singular values




D can be factorized into M and S using the Singular Value Decomposition $U_3 W_3 V_3^T$, where $W_3$ is the diagonal matrix that contains the first 3 (non-zero) singular values of the decomposition.

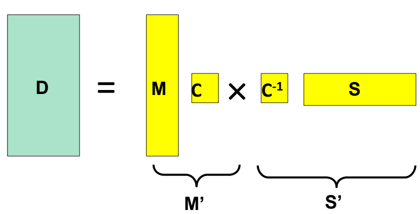
## Factorizing the Measurement Matrix by SVD




We combine $W_3$ and $V_3'$ to make the structure matrix S, and $U_3$ becomes the motion matrix M.

While this way of associating the components of the SVD decomposition to M and S leads to a physically and geometrical plausible solution of the affine SFM problem, this choice is not unique. We could also use $V_3$ to make the structure matrix S, and combine $W_3$ and $U_3$ to make the structure matrix M, since in either cases the observation matrix D is the same.

## Affine Ambiguity




- The decomposition is not unique. We get the same **D** by applying the transformations:

**M → MC**

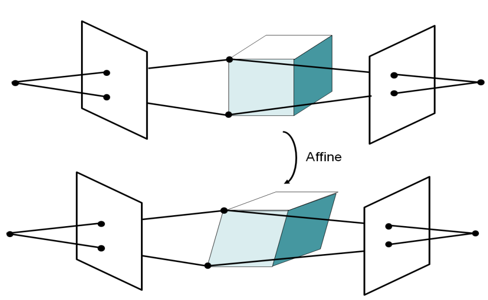
**S → $C^{-1}$S**

where **C** is an arbitrary 3x3 matrix describing an affine transformation

- Additional constraints must be enforced to resolve this ambiguity

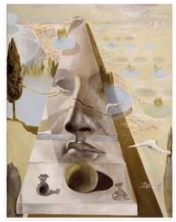
Notice that we can also arbitrarily transform the motion matrix by a 3x3 affine transform C, as long as we also transform the structure matrix by the inverse transformation $C^{-1}$: The resulting observation matrix D will still be the same. Therefore, our solution requires extra constraints to resolve this ambiguity.

## Affine Ambiguity




The slide shows an example of an affine ambiguity: if the structure is transformed into **$C^{-1}$ S**, we can always transform the camera into **M C**, such that the observations won't change.

# Lecture 7
## Multi-view geometry



- The SFM problem
- Affine SFM
- **Perspective SFM**
- Self-calibration
- Applications



Now we move to the structure from motion problem for perspective cameras.

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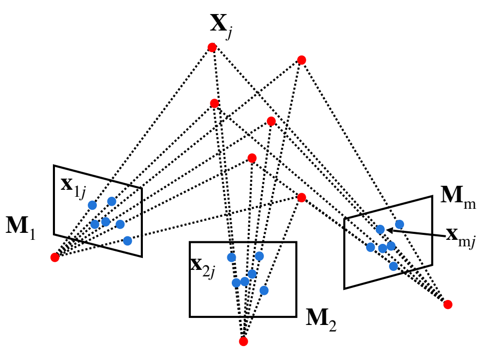
## Structure from motion problem




From the m×n correspondences $\mathbf{x}_{ij}$, estimate:

- *m* projection matrices $\mathbf{M}_i$ = motion
- *n* 3D points $\mathbf{X}_j$ = structure

Let's consider the structure from motion problem for the general case of projective cameras $\mathbf{M}_i$

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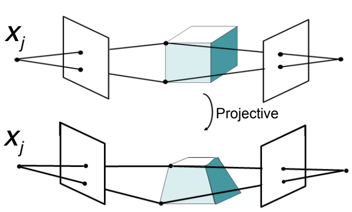
## Structure from Motion Ambiguities




- In the general case (nothing is known) the ambiguity is expressed by an arbitrary affine or projective transformation

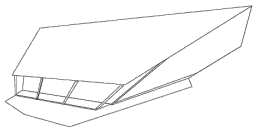
$$x_j = M_i \boxed{X_j} \rightarrow \boxed{H X_j} \qquad \boxed{M_i} = K_i \begin{bmatrix} R_i & T_i \end{bmatrix} \rightarrow \boxed{M_j H^{-1}}$$

$$x_j = M_i X_j = \left(M_i H^{-1}\right)\left(H X_j\right)$$

In this case also, solutions for structure and motion can be determined up a projective transformation: we can always arbitrarily transform the motion matrix by a 3x3 transform H, as long as we also transform the structure matrix by the inverse transformation $H^{-1}$. The resulting observations will still be the same.

## Projective Ambiguity

S =

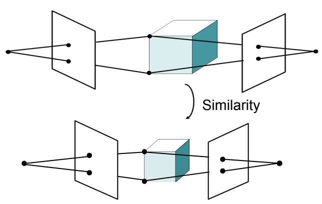


R. Hartley and A. Zisserman, Multiple View Geometry in Computer Vision, 2nd edition, 2003

This is an example of a solution of the SFM problem obtained using the two top images. This solution is correct up a projective transformation. The reconstructed S (shown in the figure) and M produce observations which are consistent with the images on top.

## Similarity Ambiguity

- The scene is determined by the images only up a similarity transformation (rotation, translation and scaling)
- This is called **metric reconstruction**



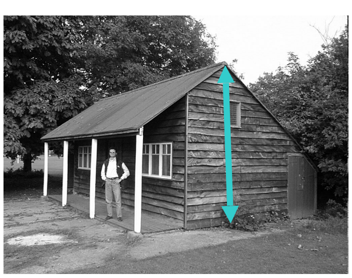

- The ambiguity exists even for (intrinsically) calibrated cameras
- For calibrated cameras, the similarity ambiguity is the only ambiguity

[Longuet-Higgins '81]

An important type of ambiguity is the similarity ambiguity – that is a reconstruction that is correct up to a similarity transform (rotation, translation and scaling). A reconstruction with only similarity ambiguity is known as a **metric reconstruction**. This ambiguity exists even when the camera are (intrinsically) calibrated. The good news is that for calibrated cameras, the similarity ambiguity is the only ambiguity [Longuet-Higgins '81].

## Similarity Ambiguity

- It is impossible, based on the images alone, to estimate the absolute scale of the scene (i.e. house height)

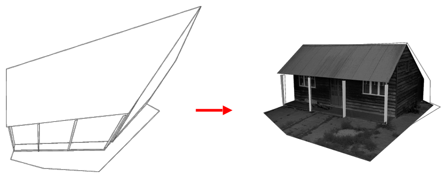


http://www.robots.ox.ac.uk/~vgg/projects/SingleView/models/hut/hutme.wrl

This is obvious when you think about it – there is no way to recover the absolute scale of a scene from images. An object's scale, absolute position and canonical orientation will always be unknown unless we make further assumptions (e.g, we know the height of the house in the figure) or incorporate more data. For instance, in the camera calibration problem we made the assumption that we know the location of the calibration points with respect to the world reference system (eg. we know the size of the squares of the checker board that we used to build the calibration rig).

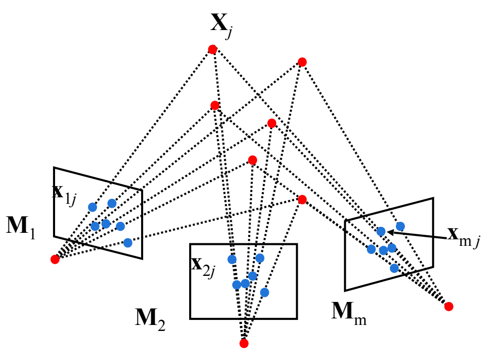
## Metric reconstruction (upgrade)

- The problem of recovering the metric reconstruction from the perspective one is called self-calibration



The problem of recovering the metric reconstruction from the perspective one is called **self-calibration** (we will discuss this in more details later). It is called metric **upgrade** in that it consists of “upgrading” a reconstruction from perspective to metric.

## Structure from motion problem




*m* cameras $M_1 \ldots M_m$

$$M_i = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & 1 \end{bmatrix}$$

In the general case of a structure from motion problem for projective cameras, we need to consider the most general camera matrix $\mathbf{M}_i$, with 11 degrees of freedom (remember the affine camera matrix only had 8).

## The Structure-from-Motion Problem

Given *m* images of *n* fixed points $X_j$ we can write

$$x_{ij} = M_i X_j \quad \text{for} \quad i = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n.$$

Problem: estimate m 3×4 matrices $M_i$ and n positions $X_j$ from m×n correspondences $x_{ij}$ .

- If the cameras are not calibrated, cameras and points can only be recovered up to a 4x4 projective (16 parameters)
- Given two cameras, how many points are needed?
- How many equations and how many unknown?

2m × n equations in 11m+3n – 16 unknowns

Similarly for the affine case, we can set up the SFM problem as the one of estimating m 3×4 matrices $M_i$ and n positions $X_j$ from m×n correspondences $x_{ij}$.

Notice that, if the cameras are not calibrated, cameras and points can only be recovered up to a 4x4 projective transformation (16 parameters): we could apply any transformation to the points as long as we applied to opposite transformation to the camera matrix. Therefore we have 11m+3n -16 unknowns in 2m x n equations. From this we can set up a counting argument for the number of views and correspondences that are required to solve for the unknowns.

## Structure-from-Motion Algorithms

- Algebraic approach (by fundamental matrix)
- Factorization method (by SVD)
- Bundle adjustment

There are several techniques for solving the general SFM problem – here we are listing a few of them. Let's start with the 1st one. Notice that all these techniques solve the SFM problem up to a perspective transformation.

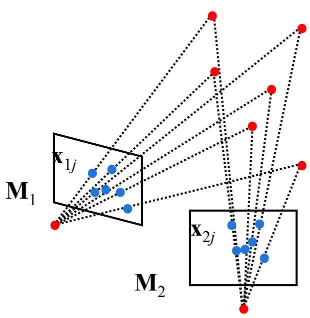
## Algebraic approach (2-view case)

1. Compute the fundamental matrix F from two views (eg. 8 point algorithm)
2. Use F to estimate projective cameras
3. Use these cameras to triangulate and estimate points in 3D

The algebraic approach leverages the concept of fundamental matrix F for solving the SFM problem for two cameras. This is the outline of the algebraic approach for the 2-camera case. First, we compute the fundamental matrix relating the points in each view. We then use F to estimate the projective camera matrices. Finally, we use these matrices to determine the world coordinates of the points (which is known up to a perspective transformation).

We already described step 1 in lecture 5. Let's now focus on step 2.

## Algebraic approach (2-view case)




$$x_{ij} = M_i X_j$$

Because of the projective ambiguity, we can always apply a projective transformation H such that:

$$M_1 H^{-1} = [I \quad 0] \qquad M_2 H^{-1} = [A \quad b]$$

[Eq. 3] Canonical perspective camera [Eq. 4]

Let's consider a camera pair as in the figure above. Since each $M_i$ can only be computed up to a perspective transformation H, we can always consider a H such that the first camera projection matrix $M_1 H^{-1}$ is canonical perspective (Eq. 3), that is, it is equal to [**I** 0]. Of course, the same transformation must also be applied to the second camera which lead to the form shown in Eq. 4.

## Algebraic approach (2-view case)

- Call **X** a generic 3D point $\mathbf{X}_{ij}$
- Call **x** and **x**' the corresponding observations to camera 1 and respectively

[Eqs. 5]
$$\begin{cases} \tilde{M}_1 = M_1 H^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \\ \tilde{M}_2 = M_2 H^{-1} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix} \\ \tilde{\mathbf{X}} = \mathrm{H}\,\mathbf{X} \end{cases}$$

$$\mathbf{x} = M_1 H^{-1} H \mathbf{X} = [\mathbf{I} \mid \mathbf{0}]\tilde{\mathbf{X}}$$
$$\mathbf{x}' = M_2 H^{-1} H \mathbf{X} = [\mathbf{A} \mid \mathbf{b}]\tilde{\mathbf{X}}$$
[Eq. 6]

$$\mathbf{x}' = [\mathbf{A} \mid \mathbf{b}]\tilde{\mathbf{X}} = [\mathbf{A} \mid \mathbf{b}]\begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \\ \tilde{X}_3 \\ 1 \end{bmatrix} = \mathbf{A}[\mathbf{I} | \mathbf{0}]\begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \\ \tilde{X}_3 \\ 1 \end{bmatrix} + \mathbf{b} = \boxed{\mathbf{A}[\mathbf{I}|\mathbf{0}]\,\tilde{\mathbf{X}}}^{?} + \mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{b}$$
[Eq. 7]

$$\mathbf{x}' \times \mathbf{b} = (\mathbf{A}\mathbf{x} + \mathbf{b}) \times \mathbf{b} = \mathbf{A}\mathbf{x} \times \mathbf{b}$$ [Eq. 8]

$$\mathbf{x}'^{T} \cdot (\mathbf{x}' \times \mathbf{b}) = \mathbf{x}'^{T} \cdot (\mathbf{A}\mathbf{x} \times \mathbf{b}) = 0$$ [Eq. 9]

$$\mathbf{x}'^{T}(\mathbf{b} \times \mathbf{A}\mathbf{x}) = 0$$ [Eq. 10]

To simplify our notation, from this point on, let's call **X** a generic 3D point $\mathbf{X}_{ij}$ and let's call **x** and **x**' the corresponding observations to camera 1 and camera 2, respectively. Since we have applied $H^{-1}$ to both camera projection matrices, we must apply H to the structure, that is, computer H X which leads to the transformed point $X_{tilde}$. The equation that describes the transformed camera projections matrices and 3D point are summarized in Eqs. 5. Using these expressions we can rewrite the observations x and x' as described in Eq. 6 (by construction, such observations won't change as we apply H to structure and motion). x' can be further manipulated so as to obtain the expression in Eq.7. Notice that we have replaced the expression of x = [I 0]$X_{tilde}$ in the last term of the derivation of Eq.7. Using this equation we can write the cross product between x' and b as shown in Eq. 8. Now, by the definition of the cross product, x' cross b is perpendicular to x', so it's dot product with x' is zero (Eq. 9). Therefore, we can determine the constraint between pairs of corresponding points shown in Eq. 10.

Se also, pag 254 HZ.

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## Cross product as matrix multiplication

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times]\mathbf{b}$$

Remember that we can represent the cross product as a matrix multiplication with the skew symmetric matrix shown in the slide. We use the square bracket and subscript x notation to denote this cross product matrix operator.

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## Algebraic approach (2-view case)

[Eqs. 5]
$$\begin{cases} \tilde{M}_1 = M_1 H^{-1} = \begin{bmatrix} I & 0 \end{bmatrix} \\ \tilde{M}_2 = M_2 H^{-1} = \begin{bmatrix} A & b \end{bmatrix} \\ \tilde{\mathbf{X}} = \mathrm{H}\,\mathbf{X} \end{cases}$$

$$\mathbf{x} = M_1 H^{-1} H \mathbf{X} = [\mathbf{I} \mid \mathbf{0}]\tilde{\mathbf{X}}$$
$$\mathbf{x}' = M_2 H^{-1} H \mathbf{X} = [\mathbf{A} \mid \mathbf{b}]\tilde{\mathbf{X}}$$
[Eq. 6]

$$\vdots$$

$$\mathbf{x}'^{T}(\mathbf{b} \times \mathbf{A}\mathbf{x}) = 0$$ [Eq. 10]

$$\mathbf{x}'^{\mathrm{T}}\boxed{[\mathbf{b}_\times]\mathbf{A}}\,\mathbf{x} = 0$$ is this familiar? $$\boxed{\mathbf{F} = [\mathbf{b}_\times]\mathbf{A}}$$

fundamental matrix!

$$\boxed{\mathbf{x}'^{\mathrm{T}}\mathrm{F}\,\mathbf{x} = 0}$$

Have we seen this constraint before? Yes – It's the fundamental matrix constraint!

# Algebraic approach (2-view case)

1. Compute the fundamental matrix F from two views (eg. 8 point algorithm)

1. $\boxed{\text{Use F to estimate projective cameras}}$
   - Compute b and A from F
   - Use b and A to estimate projective cameras

2. Use these cameras to triangulate and estimate points in 3D

As we said earlier, we assume that step 1 of the SFM algorithm is completed and we have computed the fundamental matrix. So F is known.

# Compute cameras

$$\mathbf{x}'^{T}\mathrm{F}\,\mathbf{x} = 0 \qquad \mathbf{F} = [\mathbf{b}_\times]\mathbf{A} \quad \text{[Eq. 11]}$$

## Compute **b**:

- Let's consider the product **F b**

$$\mathbf{F}\cdot\mathbf{b} = [\mathbf{b}_\times]\mathbf{A}\cdot\mathbf{b} = \mathbf{b}\times\mathbf{A}\cdot\mathbf{b} = 0 \quad \text{[Eq. 12]}$$

- Since **F** is singular, we can compute **b** as least sq. solution of **F b** = 0, with |**b**|=1 using SVD

- Using a similar derivation, we have that $\mathbf{b}^T\,\mathbf{F} = 0$

Using Eq. 11, we can decompose F and estimate A and b. Let's start by computing b. Let's consider the product F b. By using Eq. 11, we can express F b as b × A • b which must be zero. Since F is singular, b can be computed as a least square solution of F b = 0, with |b|=1, using SVD. Notice that using a similar derivation, we have that $\mathbf{b}^T$ **F** = 0

# Compute cameras

$$\mathbf{x}'^{T}\mathrm{F}\,\mathbf{x} = 0 \qquad \mathbf{F} = [\mathbf{b}_\times]\mathbf{A} \quad \text{[Eq. 11]} \qquad \begin{cases} \mathbf{F}\,\mathbf{b} = 0 \\ \mathbf{b}^T\,\mathbf{F} = 0 \end{cases}$$

## Compute **A**:

- Define: $\mathbf{A'} = -[\mathbf{b}_\times]\,\mathbf{F}$

- Let's verify that $[\mathbf{b}_\times]\mathbf{A'}$ is equal to $\mathbf{F}$ :

Indeed: $[\mathbf{b}_\times]\mathbf{A}' = -[\mathbf{b}_\times][\mathbf{b}_\times]\,\mathbf{F} = -(\mathbf{b}\,\mathbf{b}^T - |\mathbf{b}|^2\,\mathbf{I})\,\mathbf{F} = -\mathbf{b}\,\mathbf{b}^T\mathbf{F} + |\mathbf{b}|^2\,\mathbf{F} = 0 + 1\cdot\mathbf{F} = \mathbf{F}$

[Eq. 13]

- Thus, $\mathbf{A} = \mathbf{A'} = -[\mathbf{b}_\times]\,\mathbf{F}$

[Eqs. 14] $$\boxed{\tilde{M}_1 = \begin{bmatrix} I & 0 \end{bmatrix} \qquad \tilde{M}_2 = \begin{bmatrix} -[\mathbf{b}_x]\mathbf{F} & \mathbf{b} \end{bmatrix}}$$

Once b is known, we can now compute A. Let's define A' = - $[b_x]$ F and verify that A' does satisfy Eq. 11. Indeed, this is demonstrated in a number of steps in Eq. 13. In order to do so we use the property that $b^T$ F = 0 and that |b|=1.
Thus, A=A'=- $[b_x]$ F, from which we have the two expressions for the camera projections matrices in Eqs. 14.

## Interpretation of **b**

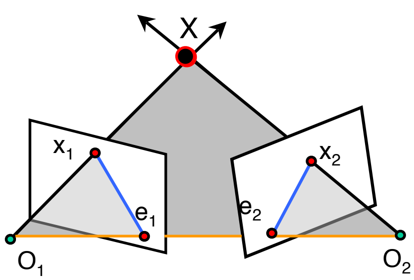
$$\mathbf{x}'^{T} F \, \mathbf{x} = 0 \qquad \mathbf{F} = [\mathbf{b}_{\times}]\mathbf{A} \qquad \begin{cases} \mathbf{F}\,\mathbf{b} = 0 & \text{[Eq. 12]} \\ \mathbf{b}^{T}\,\mathbf{F} = 0 \end{cases}$$

[Eq. 11]

What's **b**??

Before we conclude this section, we want give a geometrical interpretation for b. We know b satisfies Eq. 12.

## Epipolar Constraint [lecture 5]




$F\, x_2$ is the epipolar line associated with $x_2$ ($l_1 = F\, x_2$)

$F^T x_1$ is the epipolar line associated with $x_1$ ($l_2 = F^T x_1$)

F is singular (rank two)

$\boxed{F\, e_2 = 0 \quad \text{and} \quad F^T e_1 = 0}$

F is 3x3 matrix; 7 DOF

Remember the epipolar constraints we saw in lecture 5. The epipoles in an image were the points that mapped to zero when transformed by the fundamental matrix.

## Interpretation of **b**

$$\mathbf{x}'^{T} F \, \mathbf{x} = 0 \qquad \mathbf{F} = [\mathbf{b}_{\times}]\mathbf{A} \qquad \begin{cases} \mathbf{F}\,\mathbf{b} = 0 \\ \mathbf{b}^{T}\,\mathbf{F} = 0 \end{cases}$$

[Eq. 11]

**b** is an epipole!

$$\tilde{M}_1 = \begin{bmatrix} I & 0 \end{bmatrix} \qquad \tilde{M}_2 = \begin{bmatrix} -[\mathbf{b}_x]\mathbf{F} & \mathbf{b} \end{bmatrix}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\tilde{M}_1 = \begin{bmatrix} I & 0 \end{bmatrix} \qquad \tilde{M}_2 = \begin{bmatrix} -[\mathbf{e}_x]\mathbf{F} & \mathbf{e} \end{bmatrix}$$

[Eq. 15] [Eq. 16]

HZ, page 254
PF, page 288

We can see, therefore, that b is an epipole! This provides a new set of equations for the camera projection matrices (Eq. 15 and Eq. 16).

## Algebraic approach (2-view case)

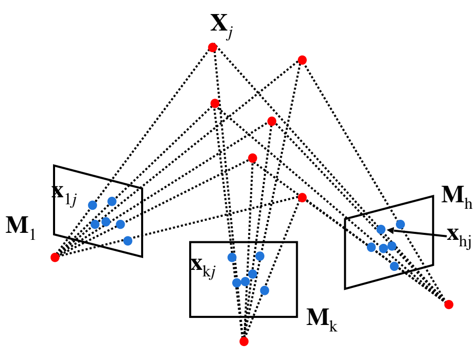
1. Compute the fundamental matrix F from two views (eg. 8 point algorithm)

1. Use F to estimate projective cameras
   - Compute b and A from F
   - Use b and A to estimate projective cameras

2. Use these cameras to triangulate and estimate points in 3D

Thus, we can now implement the third step of the approach: using the camera projection matrices $\tilde{M}_1$ and $\tilde{M}_2$ we can compute points $\tilde{X}$ in 3D from any pair of corresponding points in camera 1 and 2. Notice that the estimate of such points will be correct up the perspective transformation H ($\tilde{X}$ = H X).

## Algebraic approach for SFM problem




Solutions (motion and structure) for each image pairs $I_k$ and $I_h$

From $I_k$ and $I_h$ → $\tilde{M}_k, \tilde{M}_h, \tilde{X}_{[k,h]}$ ← 3D points associated to point correspondences available between $I_k$ and $I_h$

In summary, the algebraic approach provides solutions for the camera matrices and the 3D points for any pair of cameras $I_k$ and $I_h$ (provided that there are enough point correspondences to estimate F). The reconstructed 3D points are associated to the point correspondences available between $I_k$ and $I_h$. Notice, again, that motion and structure are metric but up a projective transformation.

Those pairwise solutions may be combined together (optimized) in a approach called *bundle adjustments* as we will see next.

## Structure-from-Motion Algorithms

- Algebraic approach (by fundamental matrix)
- Factorization method (by SVD)
- Bundle adjustment

The second method it's an adaptation from the Tomasi-Kanade method discussed in lecture 6 and follows the same principle. Unfortunately, it makes a strong assumption that relies on having available a rough estimate of distance of the points from the cameras (which may be reasonable in some applications, but not in general).

We'll now look at bundle adjustment method.

## Limitations of the approaches so far

- Factorization methods assume all points are visible.

This not true if:

  - occlusions occur
  - failure in establishing correspondences

- Algebraic methods work with 2 views

There are major limitations related to the two methods we have seen so far:

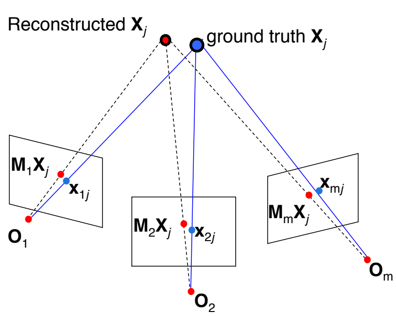
- The factorization method assumes that all points are visible in every image. This very unlikely to happen because of occlusions and failures to find correspondences.
- The algebraic approach produces pairwise solutions but not a coherent optimized reconstruction using all the cameras and 3D points.

The bundle adjustment approach addresses some of these limitations.

## Bundle adjustment

- Non-linear method for refining structure and motion
- Minimizes re-projection error

$$E(M, \mathbf{X}) = \sum_{i=1}^{m} \sum_{j=1}^{n} D\left(\mathbf{x}_{ij}, M_i \mathbf{X}_j\right)^2$$




Bundle adjustment is a non-linear method for solving the SFM problem. In the optimization we aim to minimize the re-projection error – the pixel distance between the projection of a reconstructed point into the estimated cameras (i.e., a point $\mathbf{M}_i \mathbf{X}_j$ shown in red) and its corresponding observation (i.e., $\mathbf{x}_{ij}$ shown in blue) for all the cameras and for all the points. This optimization problem is very similar to the one we introduced in lecture 5 when we talked about triangulation.

## General Calibration Problem

$$E(M, \mathbf{X}) = \sum_{i=1}^{m} \sum_{j=1}^{n} D\left(\underbrace{\mathbf{x}_{ij}}_{\text{measurements}}, \underbrace{M_i \mathbf{X}_j}_{\text{parameters}}\right)^2$$

D is the nonlinear mapping

- Newton Method
- Levenberg-Marquardt Algorithm
  - Iterative, starts from initial solution
  - May be slow if initial solution far from real solution
  - Estimated solution may be function of the initial solution
  - Newton requires the computation of J, H
  - Levenberg-Marquardt doesn't require the computation of H

As we discussed for the camera calibration problem, a common way to solve this problem is to resort to non-linear optimization techniques. Two common ones include the Newton's method and the Levenberg-Marquardt algorithm. The slide lists some of the advantages and trade-offs of Levenberg-Marquardt. J and H refer to the Jacobian and Hessian, respectively. For more details about optimization methods, please refer to [FP] Sec. 22.2 (page 669-672) or any reference text books.

## Bundle adjustment

- Advantages
  - Handle large number of views
  - Handle missing data
- Limitations
  - Large minimization problem (parameters grow with number of views)
  - Requires good initial condition

- Used as the final step of SFM (i.e., after the factorization or algebraic approach)
- Factorization or algebraic approaches provide a initial solution for optimization problem

Bundle adjustment has some important advantages and limitations when compared to the other methods we've surveyed:

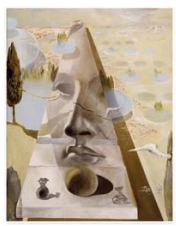
- it handles large number of views
- it handles missing data

However, it also suffers from limitations:

- it's a large minimization problem (parameters grow with number of views)
- it requires good initial condition, which we often find with one of the other methods we've discussed.

For this reason it is often used as final step of SFM (i.e., after the factorization or algebraic approach) in that a factorization or algebraic approach may provide a good initial solution for optimization problem.

# Lecture 7
## Multi-view geometry



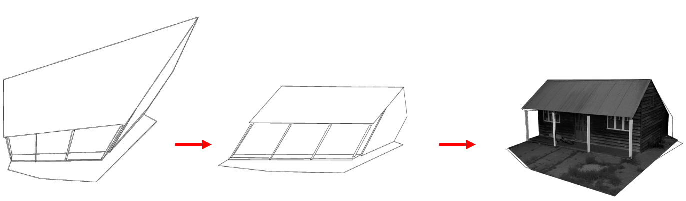
- The SFM problem
- Affine SFM
- Perspective SFM
- Self-calibration
- Applications



Now we move to the self-calibration problem

## Self-calibration

- Self-calibration is the problem of recovering the metric reconstruction from the perspective (or affine) reconstruction
- We can self-calibrate the camera by making some assumptions about the cameras



The self-calibration problem is the problem of recovering the metric reconstruction from the perspective (or affine) reconstruction. This problem is solved by assuming that certain knowledge about the cameras is available. For instance, we know the focal lengths or other intrinsic parameters of the cameras; or we know that all the images are acquired with fixed focal length.

# Self-calibration

[HZ] Chapters 19 "Auto-calibration"

Several approaches:

- Use single-view metrology constraints (lecture 4)
- $\boxed{\text{- Direct approach (Kruppa Eqs) for 2 views}}$
- Algebraic approach
- Stratified approach

There are several approaches to solve the problem of self-calibration. We have seen in lecture 4, a way of using single-view metrology constraints which can be employed for this problem. There are a few other approaches that we will study in this lecture which include:

- Direct approach (Kruppa Eqs) for 2 views
- Algebraic approach

# Direct approach

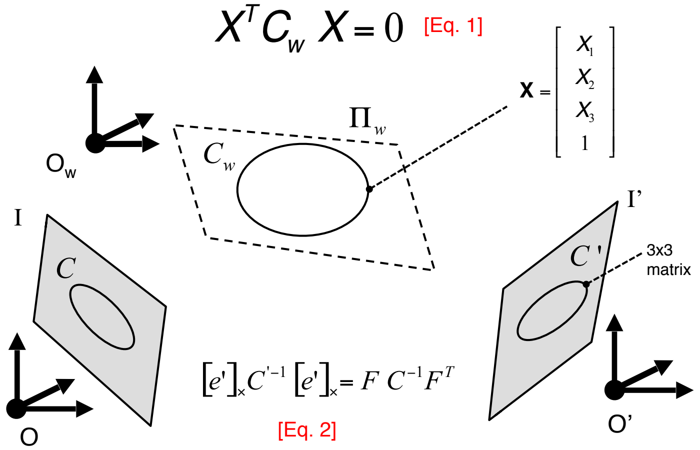
We use the following results:

1. A relationship that maps conics across views
2. Concept of absolute conic and its relationship to K
3. The Kruppa equations

In the direct approach, we use information about how a conic's image is related across different views. We introduce the notion of absolute conic and use its image (image of the absolute conic or IAC). These will lead us to the so call *Kruppa* equations.

# Projections of conics across views

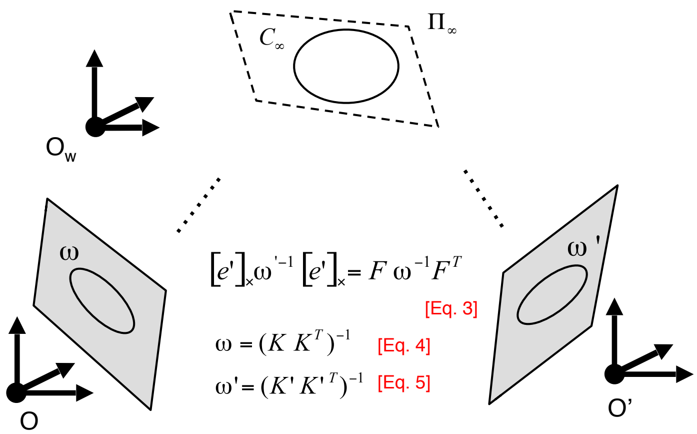
$$X^T C_w X = 0 \quad \text{[Eq. 1]}$$




Let's start by considering the conic $C_w$ which is described by a matrix $C_w$, such that points belonging to it satisfy $x^T C_w x = 0$ (which is a general $2^{nd}$ degree equation that describes a conic) (Eq. 1). X is a point in 3D in homogeneous coordinate systems. $C_w$ is imaged in two cameras (with centers at O and O') as C and C'. Eq.2 describes a relationship between C and C' through the fundamental matrix F, defined for the camera pair I and I', and the epipole e' (in second camera).

## Projection of absolute conics across views

From lecture 4, [HZ] page 210, sec. 8.5.1




$$[e']_\times \omega'^{-1} [e']_\times = F\,\omega^{-1} F^T \quad \text{[Eq. 3]}$$

$$\omega = (K\,K^T)^{-1} \quad \text{[Eq. 4]}$$

$$\omega' = (K'\,K'^T)^{-1} \quad \text{[Eq. 5]}$$

Now, remember that an absolute conic is a particular conic in the plane at infinity (lecture 5). In this slide, we consider the absolute conic and its image in the cameras with centers at O and O'. We apply Eq. 2 for the case of the absolute conic to arrive at Equation 3. From Equation 30 in Lecture 4, we have Equation 4 and Equation 5. K and K' are the camera matrices for camera 1 and 2 respectively.

## Kruppa equations

[Faugeras et al. 92] From [HZ] page 471

$$\begin{pmatrix} u_2^T K' K'^T u_2 \\ -u_1^T K' K'^T u_2 \\ u_1^T K' K'^T u_1 \end{pmatrix} \times \begin{pmatrix} \sigma_1^2 v_1^T K K^T v_1 \\ \sigma_1 \sigma_2 v_1^T K K^T v_2 \\ \sigma_2^2 v_2^T K K^T v_2 \end{pmatrix} = 0 \quad \text{[Eq. 6]}$$

where $u_i$, $v_i$ and $\sigma_i$ are the columns and singular values of SVD of F

These give us two independent constraints in the elements of K and K'

If we perform a Singular Value Decomposition of F, as $U \Sigma V'$, we can expand Equation 3 and arrive at Equation 6 (refer to HZ page 471 for the detailed derivation). In this equation, $u_1$ and $u_2$ are the left singular vectors of F, while $v_1$ and $v_2$ are the right singular vectors, and $\sigma_1$ and $\sigma_2$ are the singular values. Equation 6 gives us two independent constraints in the elements of K and K' which are called the Kruppa Equations.

## Kruppa equations

[Faugeras et al. 92]

$$\begin{pmatrix} u_2^T K' K'^T u_2 \\ -u_1^T K' K'^T u_2 \\ u_1^T K' K'^T u_1 \end{pmatrix} \times \begin{pmatrix} \sigma_1^2 v_1^T K K^T v_1 \\ \sigma_1 \sigma_2 v_1^T K K^T v_2 \\ \sigma_2^2 v_2^T K K^T v_2 \end{pmatrix} = 0$$

$$\frac{u_2^T K K^T u_2}{\sigma_1^2 v_1^T K K^T v_1} = \frac{-u_1^T K K^T u_2}{\sigma_1 \sigma_2 v_1^T K K^T v_2} = \frac{u_1^T K K^T u_1}{\sigma_2^2 v_2^T K K^T v_2} \quad \text{[Eq. 7]}$$

- Let's make the following assumption: $K' = K = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix}$ [Eq. 8]

[Eq. 9] $\alpha f^2 + \beta f + \gamma = 0 \longrightarrow f$

In the case where K and K' are equal, the Kruppa equations can be further simplified to the form in Equation 7. Moreover, if the camera matrices K and K' have the form in Equation 8, then the Kruppa equations reduce to a quadratic equation in the focal length 'f', as shown in Equation 9, from which we can compute f – this allows to estimate the intrinsic parameters of the cameras and, thus, remove the projective ambiguity.

## Kruppa equations

[Faugeras et al. 92]

- Powerful if we want to self-calibrate 2 cameras with unknown focal length
- Limitations:
  - Work on a camera pair
  - Don't work if R=0

[Eq. 10] $[e']_\times \omega^{-1} [e']_\times = F\, \omega^{-1} F^T$ becomes trivial

Since: $F = [e']_\times$

The Kruppa equations are useful when calibrating two cameras whose focal lengths are the same but are unknown (and all the other parameters are either zero or known). Some of the drawbacks are that this method can be employed only on a pair of cameras. Also, this method will not work if the relative rotation R between the cameras is 0 (pure translation motion between the two cameras). This is because Equation 10 anyway holds trivially.

## Self-calibration

[HZ] Chapters 19 "Auto-calibration"

Several approaches:

- Use single-view metrology constraints (lecture 4)
- Direct approach (Kruppa Eqs) for 2 views
- $\boxed{\text{Algebraic approach}}$
- Stratified approach

Let's now consider the algebraic approach.

## Algebraic approach Multi-view approach

Suppose we have a projective reconstruction $\{\tilde{M}_i, \tilde{X}_j\}$

Let H be a homography such that:

First perspective camera is canonical: $\tilde{M}_1 = [\; I \quad 0 \;]$ [Eq. 11]

$i^{th}$ perspective reconstruction of the camera (known): $\tilde{M}_i = [\; A_i \quad b_i \;]$ [Eq. 12]

[Eq. 13] $$(A_i - b_i p^T) K_1 K_1^T (A_i - b_i p^T)^T = K_i \; K_i^T \qquad \text{i=2...m}$$

[Eq. 14] $$H = \begin{bmatrix} K_1 & 0 \\ -p^T K_1 & 1 \end{bmatrix}$$

$p$ is an unknown 3x1 vector

$K_1 \ldots K_m$ are unknown

This approach allows working with more than just two cameras, so it is a multi-view approach. Assume that a perspective solution for the structure from motion problem (i.e. {$M_i^{tilde}$, $X_j^{tilde}$}) is available. In this approach, we first use a homographic transformation H to make one of the cameras canonical (Equation 11), similarly to what we did in the algebraic approach for solving the SFM problem (see previous slides). Thus, the other cameras will have perspective projections of the form in Equation 12. These are known in that they can be estimated by using any of the methods discussed before for the SFM problem.

By using the concept of absolute conic and plane at infinity, it is possible to derive a constraint of the form expressed in Equation 13 (see HZ pages 460-461). In such equation we have $A_i$ and $b_i$ (which are known), the camera matrices $K_i$ for i=2…m (which are unknown), the camera matrix $K_1$ (of the first camera which is also unknown), and a 3x1 vector p (which is also unknown). It can be shown that p is related to H by Equation 14 (see HZ pages 460-461).

# Algebraic approach Multi-view approach

Suppose we have a projective reconstruction

Let H be a homography such that:

First perspective camera is canonical: $\tilde{M}_1 = [\ I \quad 0\ ]$ [Eq. 11]

$i^{th}$ perspective reconstruction of the camera (known): $\tilde{M}_i = [\ A_i \quad b_i\ ]$ [Eq. 12]

[Eq. 13] $$(A_i - b_i p^T) K_1 K_1^T (A_i - b_i p^T)^T = K_i \ K_i^T \quad \text{i=2...m}$$

How many unknowns?
- 3 from $p$
- 5 m from $K_1 \ldots K_m$

How many equations? 5 independent equations [per view]

Thus, if we consider m cameras, we have in total 5×m + 3 unknowns (3 of which are in p, and 5×m are in $K_i$ matrices for i=1…m). The constraint in Eq. 13 provides 9 equations ($K_i K_i^T$ is a 3x3 matrix). However, because $K_i K_i^T$ is symmetric and defined up to scale (i.e., it has 5 independent elements, see lecture 4), it produces only 5 independent constraints. From this we can set up a counting argument for the number of views that are required to solve for the unknowns.

Once we solve our system, p and $K_1$ become known and we can compute the homographic transformation H (Eq. 14) which allows to “upgrade” the perspective solution (i.e. $\{M_i^{tilde}, X_j^{tilde}\}$) into a metric one.

# Algebraic approach Multi-view approach

Suppose we have a projective reconstruction

Let H be a homography such that:

First perspective camera is canonical: $\tilde{M}_1 = [\ I \quad 0\ ]$ [Eq. 11]

$i^{th}$ perspective reconstruction of the camera (known): $\tilde{M}_i = [\ A_i \quad b_i\ ]$ [Eq. 12]

Assume all camera matrices are identical: $K_1 = K_2 \ldots = K_m$

[Eq. 15] $$(A_i - b_i p^T) K K^T (A_i - b_i p^T)^T = K \ K^T \quad \text{i=2...m}$$

How many unknowns?
- 3 from $p$
- 5 from K

How many equations? 5 independent equations [per view]

We need at least 3 views to solve the self-calibration problem

An interesting case is the one when all the camera matrices are identical: $K_1 = K_2 \ldots = K_m = K$ and, thus, Eq. 13 becomes Eq. 15. In this case the we have 8 unknown (3 from p and 5 from K). Thus, we need at least 3 views to solve the self-calibration problem; Indeed the number of equations ( 5(m-1)) should be greater of equal to the number of unknowns (8).

# Algebraic approach

Art of self-calibration:

Use assumptions on Ks to generate enough equations on the unknowns

| *Condition* | *N. Views* |
|---|---|
| • Constant internal parameters | 3 |
| • Aspect ratio and skew known<br>• Focal length and offset vary | 4 |
| • Skew =0, all other parameters vary | 8 |

Issue: the larger is the number of view, the harder is the correspondence problem

Bundle adjustment helps!

This last example shows the general way to go about self-calibration; that is, to use various constraints on the K matrices so that we have enough equations to evaluate each of the unknowns. The table enlists the number of views needed to perform self-calibration for different conditions listed in the first column. Notice, however, as the number of views is increasing, it is difficult to establish correspondences. As we discussed earlier, in this case bundle adjustment can help.

## SFM problem - summary

1. Estimate structure and motion up perspective transformation
   1. Algebraic
   2. factorization method
   3. bundle adjustment
2. Convert from perspective to metric (self-calibration)
3. Bundle adjustment

** or **

1. Bundle adjustment with self-calibration constraints

The slide summarizes what we have discussed so far. Notice that it is always recommended to run bundle adjustment at the end of any reconstruction approach (even after the self-calibration step), to make sure that a coherent and globally optimal solution is obtained.

Another common strategy, which avoids the multiple-step process needed to recover a metric reconstruction, is to performing bundle adjustment with added self-calibration constraints.

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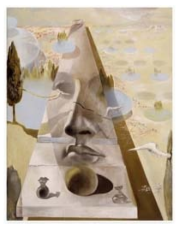
## Lecture 7

## Multi-view geometry



- The SFM problem
- Affine SFM
- Perspective SFM
- Self-calibration
- Applications

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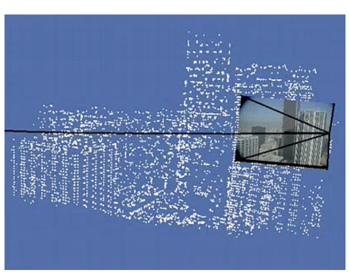
## Structure from motion problem



Courtesy of Oxford **Visual Geometry Group**

Lucas & Kanade, 81
Chen & Medioni, 92
Debevec et al., 96
Levoy & Hanrahan, 96
Fitzgibbon & Zisserman, 98
Triggs et al., 99
Pollefeys et al., 99
Kutulakos & Seitz, 99

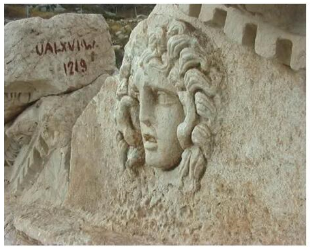
Levoy et al., 00
Hartley & Zisserman, 00
Dellaert et al., 00
Rusinkiewic et al., 02
Nistér, 04
Brown & Lowe, 04
Schindler et al, 04
Lourakis & Argyros, 04
Colombo et al. 05

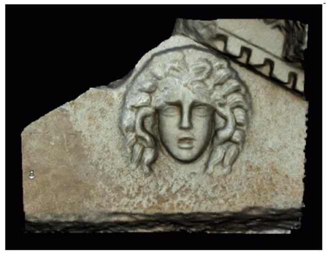
Golparvar-Fard, et al. JAEI 10
Pandey et al. IFAC , 2010
Pandey et al. ICRA 2011
Microsoft's PhotoSynth
Snavely et al., 06-08
Schindler et al., 08
Agarwal et al., 09
Frahm et al., 10

Here is we show some results from some the existing SFM pipelines.

# Reconstruction and texture mapping

M. Pollefeys et al 98---





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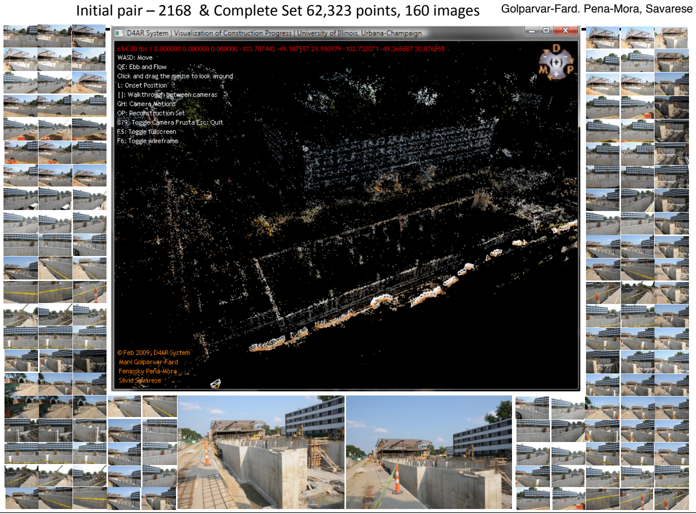
# Incremental reconstruction of construction sites

Initial pair – 2168 & Complete Set 62,323 points, 160 images

Golparvar-Fard. Pena-Mora, Savarese 2008




59

This slide animates the reconstruction. As seen the structure and motion start from 2 cameras and 2168 points and eventually it reaches to the full image dataset (160 images) and 62,323 points.

The photos are all unordered. They are taken for to monitor the progress of the construction, so there's enough photographs to recover the building's shape.

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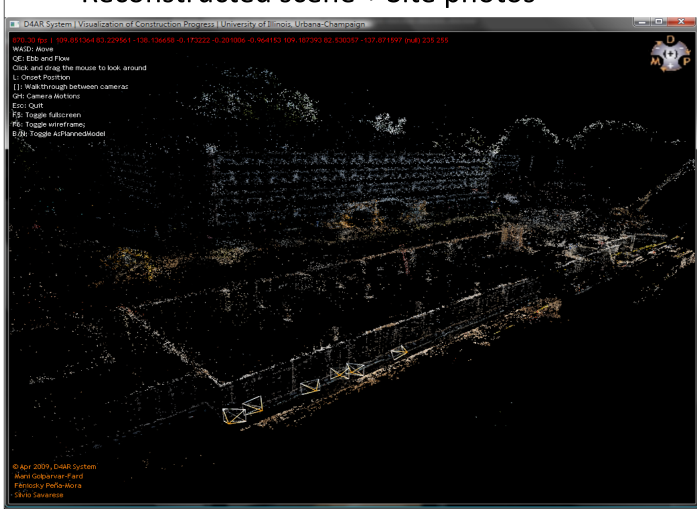
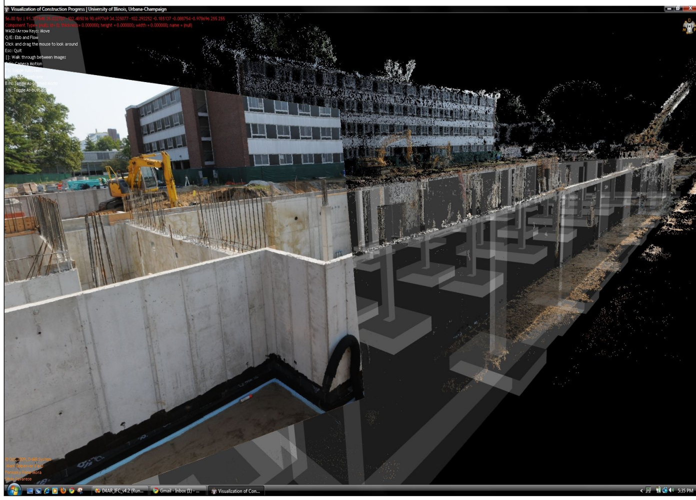
# Reconstructed scene + Site photos



## Reconstructed scene + Site photos



## Results and applications

Noah Snavely, Steven M. Seitz, Richard Szeliski, "Photo tourism: Exploring photo collections in 3D," ACM Transactions on Graphics (SIGGRAPH Proceedings),2006,

Photosynth

One particularly interest application used Flickr images from tourists to reconstruct famous landmarks such as the Colleseum.